Uniqueness of solution of a generalized $\star$-Sylvester matrix equation[☆]

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Abstract

We present necessary and sufficient conditions for the existence of a unique solution of the generalized $\star$-Sylvester matrix equation $AXB + CX^\star D = E$, where A, B, C, D, E are square matrices of the same size with real or complex entries, and where $\star$ stands for either the transpose or the conjugate transpose. This generalizes several previous uniqueness results for specific equations like the $\star$-Sylvester or the $\star$-Stein equations.

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1. Introduction

Given $A, B, C, D, E \in \mathbb{F}^{n\times n}$, with $\mathbb{F}$ being $\mathbb{C}$ or $\mathbb{R}$, we consider the equation

$$AXB + CX^\star D = E \tag{1}$$

where $X \in \mathbb{F}^{n\times n}$ is an unknown matrix, and where, for a given matrix $M \in \mathbb{F}^{n\times n}$, $M^\star$ stands for either the transpose M^T or the conjugate transpose M^*.

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In the last few years, this equation has been considered by several authors in the context of linear Sylvester-like equations arising in applications (see, for instance, [3, 14, 17, 18]).

Equation (1) is a natural extension of the $\star$-Sylvester equation

$$AX + X^\star D = E, \tag{2}$$

and it is closely related to the generalized Sylvester equation

$$AXB + CXD = E. \tag{3}$$

For this reason, we refer to (1) as a *generalized $\star$-Sylvester equation*. Note that (3) contains, as a particular case, the classical Sylvester equation

$$AX + XD = E, \tag{4}$$

in the same way as (2) is a particular case of (1).

Two of the most relevant theoretical questions regarding the solvability of these matrix equations are:

(a) Find necessary and sufficient conditions for the existence of a solution.

(b) Find necessary and sufficient conditions for the existence of a *unique* solution.

These questions can be answered when considering the matrix equation as a linear system in the entries of X (or of $\mathrm{re}(X)$ and $\mathrm{im}(X)$). However, this approach is of limited interest, since it involves matrices of much larger size and difficult to be handled. For this reason, the research efforts have been focused on getting an answer to these questions in terms of matrices or matrix pencils of the size of the matrix coefficients.

With this constraint in mind, question (a) has been already solved in the literature for all equations (1)–(4). More precisely, the characterization of consistency of the Sylvester equation (4), in terms of the matrix coefficients, was obtained back in 1952 by Roth and it is currently known as “Roth’s criterion” [15]. For the $\star$-Sylvester equation (2), a similar characterization was obtained in [19] for $\mathbb{F} = \mathbb{C}$, and later in [6] for $\mathbb{F}$ being an arbitrary field with characteristic different from 2. Recently, necessary and sufficient conditions have been obtained for the consistency of general systems containing both Sylvester and $\star$-Sylvester equations, including the case where only one type of these equations is present [8]. These systems include the case of single equations such as (1) and (3).

Regarding question (b), characterizations for the uniqueness of solutions of (2)–(4) are also known. They consist of spectral properties of matrices or matrix pencils constructed in a simple way just using the coefficient matrices. In particular, (4) has a unique solution if and only if A and D have disjoint spectrum [10, Ch. 8.1]. As for (3), it has a unique solution if and only if the pencils $A + \lambda C$ and $D - \lambda B$ are regular and have disjoint spectrum [4, Thm. 1].

The characterization of the uniqueness of solution of (2) consists of exclusion conditions on the spectrum of the pencil $A - \lambda D^\star$ [1, 13] (see Theorem 4). It is interesting to note that, for the equation $AX + CX^\star = E$, which is also a particular case of (1), the characterization of the uniqueness of solution is exactly the same as for (2) but replacing $A - \lambda D^\star$ by $A - \lambda C$ [5].

The goal of this work is to characterize the uniqueness of solution of (1) in terms of pencils with size $2n$ constructed from the coefficients A, B, C, D.

It is worth mentioning that necessary and sufficient conditions for the uniqueness of solution of (1), as a part of an algorithmic procedure, have been obtained in [3, Sec 4.2] (other iterative algorithms can be found in [16, 20], where uniqueness is not discussed). Nevertheless, these conditions are not given explicitly in terms of the coefficients and thus they do not give a satisfactory answer to question (b) for equation (1).

In summary, we can say that the characterization of the uniqueness of solution of standard Sylvester equations (3)–(4) and the characterization of uniqueness of solution of $\star$-Sylvester equations present some interesting differences. While the first one consists of exclusion conditions on the joint spectrum of a couple of two different matrix pencils constructed from the coefficient matrices, the second one consists of exclusion conditions on the spectrum of a single pencil that involves all coefficient matrices. The characterization of the uniqueness of solution of the generalized $\star$-Sylvester equation (2), proposed here, will confirm this behavior.

1.1. A pencil approach to the uniqueness problem

Equation (1) can be transformed into a linear system using: (i) the vec operator which stacks the columns of a matrix in a long vector; (ii) the Kronecker product, for which we have: $\operatorname{vec}(AXB) = (B^T \otimes A)\operatorname{vec}(X)$; and (iii) a permutation matrix Π of size $n^2 \times n^2$ such that $\Pi \operatorname{vec}(X) = \operatorname{vec}(X^T)$.

In particular:

$$\operatorname{vec}(CX^TD) = (D^T\otimes C)\operatorname{vec}(X^T) = (D^T\otimes C)\Pi\operatorname{vec}(X) = \Pi(C\otimes D^T)\operatorname{vec}(X),$$

where the last identity comes from the fact that the similarity through Π inverts the order of the Kronecker product [11, Sec. 4.3].

As a consequence, equation (1) is equivalent to the system

$$\big((B^T\otimes A)+\Pi(C\otimes D^T)\big)\operatorname{vec}(X) = \operatorname{vec}(E,) \quad \text{if } \star = T, \tag{5}$$

$$(B^T\otimes A)\operatorname{vec}(X)+\Pi(C\otimes D^T)\operatorname{vec}(\overline{X}) = \operatorname{vec}(E) \quad \text{if } \star = *, \tag{6}$$

where $\overline{X}$ denotes the conjugate of the matrix X. The system of equations (5) is linear over $\mathbb{C}$, but the system (6) is not. However, if we split the real and imaginary parts of all coefficient matrices A, B, C, D, E, as well as the real and imaginary parts of the unknown matrix X, then (6) is equivalent to the linear system

$$R\operatorname{vec}\big(\begin{bmatrix} \operatorname{re}(X) & \operatorname{im}(X) \end{bmatrix}\big) = \operatorname{vec}\big(\begin{bmatrix} \operatorname{re}(E) & \operatorname{im}(E) \end{bmatrix}\big), \tag{7}$$

for some matrix R with real entries and size $(2n^2)\times(2n^2)$.

As a first approach to address the uniqueness of solution of (1) note that the maps

$$\begin{array}{cccc} \mathcal{F}: & \mathbb{C}^{n^2} & \longrightarrow & \mathbb{C}^{n^2} \\ & \operatorname{vec}(X) & \mapsto & \big((B^T\otimes A)+\Pi(C\otimes D^T)\big)\operatorname{vec}(X), \end{array}$$

and

$$\begin{array}{cccc} \mathcal{R}: & \mathbb{R}^{2n^2} & \longrightarrow & \mathbb{R}^{2n^2} \\ & \operatorname{vec}\big(\begin{bmatrix} \operatorname{re}(X) & \operatorname{im}(X) \end{bmatrix}\big) & \mapsto & R\operatorname{vec}\big(\begin{bmatrix} \operatorname{re}(X) & \operatorname{im}(X) \end{bmatrix}\big), \end{array}$$

with R as in (7), are linear maps. Then (1) has a unique solution, for any right hand side $E\in\mathbb{C}^{n\times n}$, if and only if the homogeneous equation

$$AXB + CX^\star D = 0 \tag{8}$$

has only the trivial solution. As a consequence, for the uniqueness of the solution we can focus on equation (8) instead of (1).

From the point of view of applications, the most interesting situation of the case $\star = T$ is when all coefficient matrices are real, so that the map $\mathcal{F}$ above can be seen as a real map and the solution X we are looking for is real as well. Also, when all coefficient matrices A, B, C, D, E have real entries, the unique solution of (1) with $\star = *$ has real entries as well. To see this, note that, by taking conjugates in (1), also $\overline{X}$ is a solution of (1), so it must be $X = \overline{X}$, hence X must be real. Then, in the case $\mathbb{F} = \mathbb{R}$, we can only consider $\star = T$. However, we cover the more general situations for the sake of completeness and the ease of statements.

Unfortunately, neither the system (5) nor the system (7) can be readily used to study equation (1), since the coefficient matrices $(B^T \otimes A) + \Pi(C \otimes D^T)$ and R, respectively, are not easy to handle.

The main result of this paper is a characterization of the uniqueness of solution of (1) in terms of elementary spectral properties of the matrix pencil $\left[\begin{smallmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{smallmatrix}\right]$. These properties are exclusion conditions of eigenvalues, similar to the ones given for the pencil $A - \lambda D^\star$ in the case of the $\star$-Sylvester equation (2). In particular, our conditions contain the characterization for the uniqueness of solution for equation (2), as well as the one for the equation $AX + CX^\star = E$, considered in [5]. Another relevant instance of (1) is the $\star$-Stein equation $AXB + X^\star = E$, which has been recently considered in the literature [2, 3, 9, 12]. Our results allow one to derive, as a particular case, the conditions for uniqueness obtained in [12, Thm. 3] and [2, Thm. 4] for this equation and $\star = T$.

The paper is organized as follows. In Section 2 we introduce or recall the basic notions and definitions used along the paper, and we state some basic results which are used later. In Section 3 we present a couple of previous results (the characterization of the uniqueness of solution of $\star$-Sylvester and $\star$-Stein equations), together with some technical results which allow us to reduce the proof of the main theorems to the case of $\star$-Sylvester equations. The main result (namely, Theorem 15) is presented, and proved, in Section 4. Finally, in Section 5 we summarize the contributions of the paper and we indicate a natural continuation of this work.

2. Notation, definitions, and basic results

Given a matrix $M \in \mathbb{C}^{n \times n}$, $M^{-\star}$ denotes the inverse of $M^\star$. The notation I stands for the identity matrix of size $n \times n$.

Our main result is a characterization of the uniqueness of solution of equation (8) in terms of spectral properties of a matrix pencil constructed from the coefficients A, B, C, D. We recall here some standard notation and results from the theory of matrix pencils that will be used along the paper. For more information on this topic, we refer the reader to [10, Ch. XII].

A matrix pencil $P(\lambda) = M - \lambda N$, with $M, N \in \mathbb{C}^{n\times n}$ is said to be *regular* if $\det(P(\lambda))$ is not identically zero. Otherwise, the pencil is said to be *singular*. A *finite eigenvalue* of a regular matrix pencil $P(\lambda)$ is a number $\lambda_0 \in \mathbb{C}$ such that $\det(P(\lambda_0)) = 0$. The regular pencil $M - \lambda N$ has an *infinite eigenvalue* if $N - \lambda M$ has 0 as eigenvalue (equivalently, if N is singular). In particular, the eigenvalues of a matrix M coincide with the eigenvalues of the pencil $M - \lambda I$. The *spectrum* of a regular matrix pencil $P(\lambda)$, denoted by $\Lambda(P)$, is the set of eigenvalues of $P(\lambda)$ (finite and infinite). Analogously, the spectrum of the matrix M is denoted by $\Lambda(M)$. The *algebraic multiplicity* of an eigenvalue λ_0 of $P(\lambda)$ is the multiplicity of λ_0 as a root of the polynomial $\det(P(\lambda))$, while the multiplicity of the infinite eigenvalue is $n - \deg(\det(P(\lambda)))$.

A *strictly equivalent* pencil to $P(\lambda)$ is a pencil of the form $UP(\lambda)V$, with $U, V \in \mathbb{C}^{n\times n}$ invertible. Accordingly, the relation on the set of matrix pencils obtained by multiplying a given pencil on the left and/or the right by invertible matrices is called *strict equivalence*. Two strictly equivalent matrix pencils have the same eigenvalues (finite and infinite) with the same algebraic multiplicity. An eigenvalue of a pencil or a matrix is *simple* if it has algebraic multiplicity equal to 1. The algebraic multiplicity of an eigenvalue λ_0 of a pencil $P(\lambda)$ or a matrix M will be denoted by $m_{\lambda_0}(P)$ or $m_{\lambda_0}(M)$, respectively.

The characterization of the uniqueness of solution of equation (8) will strongly depend on the following notion, where we consider the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the conventions $0^{-1} = \infty$, $\infty^{-1} = 0$, $\overline{\infty} = \infty$.

Definition 1. (Reciprocal free and $*$-reciprocal free set) [13]. *Let $\mathcal{S}$ be a subset of $\overline{\mathbb{C}}$. We say that $\mathcal{S}$ is*

(a) reciprocal free *if $\lambda \neq \mu^{-1}$, for all $\lambda, \mu \in \mathcal{S}$;*

(b) $*$-reciprocal free *if $\lambda \neq (\overline{\mu})^{-1}$, for all $\lambda, \mu \in \mathcal{S}$.*

This definition includes the values $\lambda = 0, \infty$, since for these values $\lambda^{-1} = (\overline{\lambda})^{-1} = \infty, 0$, respectively.

Remark 1. Note that if $\mathcal{S} \subseteq \overline{\mathbb{C}}$ is reciprocal free, then $\pm 1 \notin \mathcal{S}$, since λ, μ in Definition 1 can be equal. Similarly, if $\mathcal{S} \subseteq \overline{\mathbb{C}}$ is $*$-reciprocal free, then $\mathcal{S}$ cannot contain any number of modulus 1.

In the following, the name $\star$-*reciprocal free* set stands for both reciprocal and $*$-reciprocal free sets. We will use the following basic results on $\star$-reciprocal free sets.

Lemma 2. *Let $M, N \in \mathbb{C}^{n\times n}$. Then $\Lambda(M - \lambda N)$ is $\star$-reciprocal free if and only if $\Lambda(N - \lambda M)$ is $\star$-reciprocal free.*

Proof. The result is an immediate consequence of the fact that $\lambda_0 \in \Lambda(M - \lambda N)$ if and only if $\lambda_0^{-1} \in \Lambda(N - \lambda M)$ (including $\lambda_0 = 0, \infty$). $\square$

Lemma 3. *Let $\mathcal{S}$ be a subset of $\overline{\mathbb{C}}$, and define*

$$\sqrt{\mathcal{S}} := \{z \in \overline{\mathbb{C}} : \ z^2 \in \mathcal{S}\},$$

with the convention $\infty^2 = \infty$. Then $\mathcal{S}$ is $\star$-reciprocal free if and only if $\sqrt{\mathcal{S}}$ is $\star$-reciprocal free.

Proof. Let us first assume that $\mathcal{S}$ is reciprocal free, and let $z \in \sqrt{\mathcal{S}}$. Then $z^2 = s \in \mathcal{S}$. Now, if $1/z \in \sqrt{\mathcal{S}}$, we would have $1/s = (1/z)^2 \in \mathcal{S}$, which is a contradiction with the fact that $\mathcal{S}$ is reciprocal free. Conversely, if $\sqrt{\mathcal{S}}$ is reciprocal free, let $s \in \mathcal{S}$, so that $s = z^2$, for some $z \in \sqrt{\mathcal{S}}$. If $1/s \in \mathcal{S}$, then we would have $(1/z)^2 = 1/s$, so that $1/z \in \sqrt{\mathcal{S}}$, which is a contradiction with the fact that $\sqrt{\mathcal{S}}$ is reciprocal free.

For $*$-reciprocal free sets the proof is analogous. Let us first assume that $\mathcal{S}$ is $*$-reciprocal free, and let $z \in \sqrt{\mathcal{S}}$, so that $z^2 = s \in \mathcal{S}$. Now, if $1/\overline{z} \in \sqrt{\mathcal{S}}$, we would have $1/\overline{s} = (1/\overline{z})^2 \in S$, a contradiction. Conversely, if $\sqrt{\mathcal{S}}$ is $*$-reciprocal free, let $s \in \mathcal{S}$, so that $s = z^2$, for some $z \in \sqrt{\mathcal{S}}$. If $1/\overline{s} \in \mathcal{S}$, then we would have $(1/\overline{z})^2 = 1/\overline{s}$, so that $1/\overline{z} \in \sqrt{\mathcal{S}}$, again a contradiction.

Notice that these arguments hold also when $\mathcal{S}$ contains 0 and/or ∞. $\square$

3. Reduction process and auxiliary results

In this section, we show that the problem of the uniqueness of solution of the general equation (8) can be reduced to the analysis of simpler equations of the same type. We also present some technical results that will be used in Section 4.

For the sake of completeness, and for further reference, let us recall the characterization of uniqueness of solution of $\star$-Sylvester equations.

Theorem 4. (Characterization of uniqueness of solution of $\star$-Sylvester equations [1, 13]). *Let $A, D \in \mathbb{C}^{n\times n}$. Then the matrix equation $AX + X^\star D = 0$ has a unique solution if and only if the pencil $A - \lambda D^\star$ is regular and:*

- *If* $\star = T$, $\Lambda(A - \lambda D^T) \setminus \{1\}$ *is reciprocal free and* $m_1(A - \lambda D^T) \leqslant 1$.
- *If* $\star = *$, $\Lambda(A - \lambda D^\star)$ *is* $*$*-reciprocal free.*

The proof of Theorem 4 for the case $\star = *$ in [13] relies on some continuity arguments. For a different proof of Theorem 4 relying only on matrix manipulations, see [7, Thms. 10-11].

If the matrices A and C in equation (8) are both singular, then there exist nonzero vectors v, w such that $Av = Cw = 0$, and thus the nonzero matrix $vw^\star$ is a solution of (8). Also, if B and D are both singular, then there exist nonzero vectors v, w such that $B^\star v = D^\star w = 0$, and thus the nonzero matrix $wv^\star$ is a solution of (8). We have thus proved the following result.

Lemma 5. *The following two conditions are necessary for the existence of a unique solution to equation* (8)*:*

(i) *At least one of the matrices* A *and* C *is invertible.*

(ii) *At least one of the matrices* B *and* D *is invertible.*

Later, in Theorem 15, we will show that the characterization of the uniqueness of solution of (8) depends on some spectral properties of the matrix pencil

$$Q(\lambda) = \begin{bmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{bmatrix}. \tag{9}$$

The following technical result deals with the determinant of this pencil when at least one of the coefficients A and C is invertible.

Lemma 6. *Let* $A, B, C, D \in \mathbb{C}^{n \times n}$ *and let* $Q(\lambda)$ *be the pencil in* (9)*.*

(a) *If* A *is invertible, then*

$$\det(Q(\lambda)) = \pm \det(A) \det(B^\star - \lambda^2 D^\star A^{-1} C).$$

(b) *If* C *is invertible, then*

$$\det(Q(\lambda)) = \pm \det(C) \det(B^\star C^{-1} A - \lambda^2 D^\star).$$

Proof. (a) When A is invertible,

$$\begin{bmatrix} 0 & I \\ I & \lambda D^\star A^{-1} \end{bmatrix} \begin{bmatrix} -\lambda D^* & B^\star \\ A & -\lambda C \end{bmatrix} = \begin{bmatrix} A & -\lambda C \\ 0 & B^\star - \lambda^2 D^\star A^{-1} C \end{bmatrix}.$$

Taking determinants we get

$$(-1)^n \det(Q(\lambda)) = \det(A) \det(B^\star - \lambda^2 D^\star A^{-1} C),$$

which gives the result.

(b) When C is invertible, the following identity holds

$$\begin{bmatrix} \lambda I & B^\star C^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{bmatrix} = \begin{bmatrix} -\lambda^2 D^\star + B^\star C^{-1} A & 0 \\ A & -\lambda C \end{bmatrix}.$$

Taking determinants in both sides, we get

$$\lambda^n \det(Q(\lambda)) = \det(B^\star C^{-1} A - \lambda^2 D^\star)(-\lambda)^n \det(C),$$

and this completes the proof. □

The following result also deals with spectral properties of the pencil $Q(\lambda)$.

Lemma 7. *Let the matrix pencil $Q(\lambda)$ in (9) be regular. Then the values 0 and ∞ are eigenvalues of $Q(\lambda)$ if and only if AB and CD are singular.*

Proof. The matrix pencil $Q(\lambda)$ has 0 as an eigenvalue if and only if the matrix $\left[\begin{smallmatrix} 0 & B^\star \\ A & 0 \end{smallmatrix}\right]$ is singular and this happens if and only if one of the matrices A or B (and thus AB) is singular. On the other hand, $Q(\lambda)$ has ∞ as an eigenvalue if and only if the matrix $\left[\begin{smallmatrix} -D^\star & 0 \\ 0 & -C \end{smallmatrix}\right]$ is singular and this happens if and only if one of the matrices C or D (and thus CD) is singular. □

Another spectral property of the pencil $Q(\lambda)$ in (9) is given in Lemma 8. This property is not going to be explicitly used in Section 4, but is implicitly used in some arguments and claims.

Lemma 8. *Assume that the matrix pencil $Q(\lambda)$ in (9) is regular. Then $\lambda_0 \in \Lambda(Q)$ if and only if $-\lambda_0 \in \Lambda(Q)$ and, moreover, the partial multiplicities of both λ_0 and $-\lambda_0$ in $Q(\lambda)$ coincide. In particular, λ_0 is a simple eigenvalue of $Q(\lambda)$ if and only if $-\lambda_0$ is a simple eigenvalue of $Q(\lambda)$.*

Proof. Since
$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} Q(\lambda) \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} = Q(-\lambda),$$
we have that the two pencils $Q(\lambda)$ and $Q(-\lambda)$ are strictly equivalent and thus they have the same eigenvalues with the same partial multiplicities. $\square$

Our reduction process will end up with $\star$-Sylvester equations, but going through $\star$-Stein equations. For this, we need the following result that relates the solution of the $\star$-Stein equation $AXB + X^\star = 0$ with the solution of an associated $\star$-Sylvester equation.

Lemma 9. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the equation $AXB + X^\star = 0$ has a unique solution if and only if the equation $AB^\star Y + Y^\star = 0$ has a unique solution.*

Proof. First assume that there is some $X \neq 0$ such that $AXB + X^\star = 0$. Applying the $\star$ operator and premultiplying by A, we get $AB^\star(X^\star A^\star) + AX = 0$, so that $Y = (AX)^\star$ is a solution of $AB^\star Y + Y^\star = 0$. It remains to prove that Y is nonzero. By contradiction, $Y = 0$ implies $AX = 0$ and, since $AXB + X^\star = 0$, this would imply $X = 0$.

The opposite direction can be proved in a similary way. If we assume that $Y \neq 0$ is such that $AB^\star Y + Y^\star = 0$, then we get that $X = B^\star Y$ is a nonzero solution of $AXB + X^\star = 0$. $\square$

Lemma 9 has a remarkable consequence.

Theorem 10. (Characterization of uniqueness of solution of $\star$-Stein equations). *Let $A, B \in \mathbb{C}^{n\times n}$. Then the equation $AXB + X^\star = 0$ has a unique solution if and only if the following conditions hold:*

- *If $\star = T$, $\Lambda(AB^T) \setminus \{1\}$ is reciprocal free and $m_{AB^T}(1) \leqslant 1$.*
- *If $\star = *$, $\Lambda(AB^*)$ is $*$-reciprocal free.*

The case $\star = T$ of Corollary 10 has been already obtained in [12, Thm. 3] and [2, Thm. 4] by different means.

The following result is key in the reduction process, since it relates the uniqueness of solution of (8) with the uniqueness of solution of two associated $\star$-Sylvester equations.

Theorem 11. *The equation $AXB+CX^\star D=0$ has a unique solution if and only if at least one of the following conditions* (a)–(b) *holds:*

(a) *A is invertible and $D^\star A^{-1}CY+Y^\star B=0$ has a unique solution;*

(b) *C is invertible and $B^\star C^{-1}AY+Y^\star D=0$ has a unique solution.*

Proof. Let us first assume that (8) has a unique solution. Then by Lemma 5 we know that at least one between A and C is invertible and at least one between B and D is invertible. Therefore, we consider the following two cases.

Case 1: A invertible. Proceeding by contradiction, we assume that $D^\star A^{-1}CY+Y^\star B=0$ has a solution $Y\neq 0$. If D is invertible, then we have $C(YD^{-1})D+A(D^{-\star}Y^\star)B=0$, thus $D^{-\star}Y^\star$ is a nonzero solution of (8), a contradiction. If B is invertible, then the $\star$-Stein equation $D^\star A^{-1}CYB^{-1}+Y^\star=0$ has a nonzero solution. Lemma 9 implies that the equation $D^\star A^{-1}CB^{-\star}Y+Y^\star=0$ has a nonzero solution and, by Lemma 9 again, the equation $D^\star ZB^{-1}C^\star A^{-\star}+Z^\star=0$ has a solution $Z\neq 0$. Then $D^\star(ZB^{-1})C^\star+B^\star(B^{-\star}Z^\star)A^\star=0$, which means that ZB^{-1} is a nonzero solution of (8), again a contradiction.

Case 2: C invertible. The proof is obtained by applying Case 1 to the equation $CXD+AX^\star B=0$.

To prove the converse, we assume first that (a) is true, in particular A is invertible. Let us assume, by contradiction that (8) has a solution $X\neq 0$, which is also a solution of $(D^\star X)B+D^\star A^{-1}C(X^\star D)=0$, and thus $Y=X^\star D$ is a solution of $D^\star A^{-1}CY+Y^\star B=0$. Then $X^\star D=0$ and, since X is a solution of (8), this implies $XB=0$. But, since at least one between B and D is invertible, it must be $X=0$, a contradiction.

The case where (b) is assumed to be true is similar. □

As an immediate consequence of Theorems 4 and 11 we get the following.

Corollary 12. *Equation* (8) *has a unique solution if and only if at least one of the following conditions* (a)–(b) *holds:*

(a) *A is invertible, $P(\lambda)=B^\star-\lambda D^\star A^{-1}C$ is regular, and it satisfies:*

  (a1) *If $\star=T$, $\Lambda(P)\setminus\{1\}$ is reciprocal free and $m_1(P)\leqslant 1$;*

  (a2) *if $\star=*$, $\Lambda(P)$ is $*$-reciprocal free.*

(b) *C is invertible, $P(\lambda) = D^\star - \lambda B^\star C^{-1} A$ is regular, and it satisfies:*

(b1) *If $\star = T$, $\Lambda(P) \setminus \{1\}$ is reciprocal free, and $m_1(P) \leqslant 1$;*

(b2) *if $\star = *$, $\Lambda(P)$ is $*$-reciprocal free.*

We emphasize that Corollary 12 generalizes the known results for equations (2) (given in Theorem 4) and $AX + CX^\star = 0$ (given in [5, Th. 5.2]).

Below we give the counterparts of Theorem 11 and Corollary 12, obtained after Corollary 5 replacing the roles of A and D by C and B, respectively.

Theorem 13. *The equation $AXB + CX^\star D = 0$ has a unique solution if and only if at least one of the following conditions* (a)–(b) *holds:*

(a) *B is invertible and $AY + Y^\star DB^{-1}C^\star = 0$ has a unique solution;*

(b) *D is invertible and $CY + Y^\star BD^{-1}A^\star = 0$ has a unique solution.*

Corollary 14. *Equation* (8) *has a unique solution if and only if at least one of the following conditions* (a)–(b) *holds:*

(a) *B is invertible, $P(\lambda) = A - \lambda C B^{-\star} D^\star$ is regular, and it satisfies:*

(a1) *If $\star = T$, $\Lambda(P) \setminus \{1\}$ is reciprocal free and $m_1(P) \leqslant 1$;*

(a2) *if $\star = *$, $\Lambda(P)$ is $*$-reciprocal free.*

(b) *D is invertible, $P(\lambda) = C - \lambda A D^{-\star} B^\star$ is regular, and it satisfies:*

(b1) *If $\star = T$, $\Lambda(P) \setminus \{1\}$ is reciprocal free, and $m_1(P) \leqslant 1$;*

(b2) *if $\star = *$, $\Lambda(P)$ is $*$-reciprocal free.*

4. Characterization of the uniqueness of solution

In this section we state and prove the main result of this paper, namely, the characterization of the uniqueness of solution of (8), which is given in the following result.

Theorem 15. *Let $A, B, C, D \in \mathbb{C}^{n\times n}$, and let $Q(\lambda) = \left[\begin{smallmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{smallmatrix}\right]$. Then the equation $AXB + CX^\star D = 0$ has a unique solution if and only if $Q(\lambda)$ is regular and:*

- *If $\star = T$, $\Lambda(Q) \setminus \{\pm 1\}$ is reciprocal free and $m_1(Q) = m_{-1}(Q) \leqslant 1$.*

- *If $\star = *$, $\Lambda(Q)$ is $*$-reciprocal free.*

Proof. We first consider the case $\star = T$.

Let us first assume that (8) has a unique solution. Then, at least one of (a) or (b) in the statement of Corollary 12 occur.

Let us focus first on case (a). Set $Q_1(\lambda) = B^\star - \lambda D^\star A^{-1}C$. By Lemma 6–(a), we have $\Lambda(Q) = \sqrt{\Lambda(Q_1)}$ and $m_1(Q) = m_{-1}(Q) = m_1(Q_1)$. Since

$$\sqrt{\mathcal{S} \setminus \{1\}} = \sqrt{\mathcal{S}} \setminus \{\pm 1\},$$

for any $\mathcal{S} \subseteq \overline{\mathbb{C}}$, then (a1) in the statement of Corollary 12, together with Lemma 3, imply that $\Lambda(Q) \setminus \{\pm 1\}$ is reciprocal free and $m_1(Q) = m_{-1}(Q) = m_1(Q_1) \leqslant 1$, as wanted.

For case (b) in Corollary 12 the arguments are exactly the same replacing $Q_1(\lambda)$ by $Q_2(\lambda) = D^\star - \lambda B^\star C^{-1} A$.

For the converse, assume that the conditions in the statement about $Q(\lambda)$ hold. Since $\Lambda(Q) \setminus \{\pm 1\}$ is reciprocal free, by Lemma 7, one of AB or CD is invertible. Then A is invertible or D is invertible. In the first case, and reversing the arguments above for the converse implication, we conclude that $Q_1(\lambda)$ is regular, $\Lambda(Q_1) \setminus \{1\}$ is reciprocal free, and $m_1(Q_1) \leqslant 1$, and then Corollary 12 implies that (8) has a unique solution. In the second case, the arguments are the same with $Q_2(\lambda)$ instead of $Q_1(\lambda)$.

The proof for $\star = *$ mimics the one for $\star = T$, just replacing the transpose with the conjugate transpose and removing the condition on the eigenvalues $\lambda = \pm 1$ along the proof. $\square$

Observing that, with respect to uniqueness, the equation $AXB + CX^\star D = 0$ is equivalent to other $\star$-Sylvester equations, such as

$$D^\star X C^\star + B^\star X^\star A^\star = 0, \qquad CXD + AX^\star B = 0, \qquad B^\star X A^\star + D^\star X^\star C^\star = 0,$$

we could replace $Q(\lambda)$ in the statement of Theorem 15 by any of the following pencils

$$\begin{bmatrix} -\lambda A & C \\ D^\star & -\lambda B^\star \end{bmatrix}, \qquad \begin{bmatrix} -\lambda B^\star & D^\star \\ C & -\lambda A \end{bmatrix}, \qquad \begin{bmatrix} -\lambda C & A \\ B^\star & -\lambda D^\star \end{bmatrix}.$$

Other variations can be obtained by changing both signs of A and B or C and D, or by changing the role of the leading and trailing coefficient in $Q(\lambda)$ (see Lemma 2). Thus, Theorem 15 can be stated using, for instance,

$$\begin{bmatrix} \lambda D^\star & B^\star \\ A & \lambda C \end{bmatrix}, \qquad \begin{bmatrix} \lambda D^\star & -B^\star \\ -A & \lambda C \end{bmatrix}, \qquad \begin{bmatrix} -D^\star & \lambda B^\star \\ \lambda A & -C \end{bmatrix}.$$

5. Conclusions and open problems

We have obtained necessary and sufficient conditions for the uniqueness of solution of the matrix equation $AXB + CX^\star D = E$, for any right hand side, explicitly in terms of the coefficients A, B, C, D. More precisely, these conditions are given in terms of spectral properties of the pencil $\left[\begin{smallmatrix} -\lambda D^\star & B^\star \\ A & -\lambda C \end{smallmatrix}\right]$ and are very simple to state. Our characterization includes, as particular cases, the ones already known for the $\star$-Sylvester equation $AX + X^\star D = E$, the $\star$-Stein equation $AXB + X^\star = E$, and the equation $AX + CX^\star = E$. In view of possible applications, a subject of future work is the design and numerical analysis of an efficient algorithm for the solution of this equation.

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